

The Distributional Zeta-Function in Disordered Field Theory

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Abstract

In this paper we present a new mathematical rigorous technique for computing the average free energy of a disordered system with quenched randomness, using the replicas. The basic tool of this technique is a distributional zeta-function, a complex function whose derivative at the origin yields the average free energy of the system as the sum of two contributions: the first one is a series in which all the integer moments of the partition function of the model contribute; the second one, which can not be written as a series of the integer moments, can be made as small as desired. This result supports the use of integer moments of the partition function, computed via replicas, for expressing the average free energy of the system. One advantage of the proposed formalism is that it does not require the understanding of the properties of the permutation group when the number of replicas goes to zero. Moreover, the symmetry is broken using the saddle-point equations of the model. As an application for the distributional zeta-function technique, we obtain the average free energy of the disordered $\lambda\varphi^4$ model defined in a d -dimensional Euclidean space.

KEYWORDS: disordered systems; average free-energy; replicas; distributional zeta-function.

PACS numbers: 05.20.-y, 75.10.Nr

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1 Introduction

Disordered systems have been investigated for decades in statistical mechanics [1–5], condensed matter, gravitational physics [6–11] and even number theory [12]. The physics of spin glasses, disordered electronic systems and directed polymers in random media are well known examples of such systems. In statistical mechanics, the disorder may significantly affect critical behavior of different systems [13]. An example of a system with disorder is the random field Ising model [14], whose hamiltonian is analogous to the one of the classical Ising model, but allowing for a quenched random magnetic field. This model can be analyzed using a continuous description, instead of dealing with discrete elements. The continuous model is obtained in the limit of zero lattice spacing. In this case, the hamiltonian of the discrete model can be replaced by an effective hamiltonian of the Landau-Ginzburg model, where the order parameter is a continuous field coupled with a quenched random field. This continuous model has been extensively studied using statistical field theory methods (e.g., [15–23]). The main problem of this disordered system is that a given realization of disorder is not invariant by translation and the correlation functions are not similar to the ones of translationally invariant systems. Also, in the presence of disorder, ground state configurations of the scalar field are defined by a saddle-point equations and the solutions of such equation depend on particular configurations of the random field. Due to the presence of these local minima, it is very difficult to implement a perturbative approach in a straightforward way.

An established procedure to circumvent the above discussed problems is to average some thermodynamic extensive quantity with respect to the probability distribution of the disorder. From a disordered system, where all the correlation functions are not translational invariant, averaging on the disorder one obtains a system where the correlation functions are translational invariant. However, this procedure is not always straightforward. For quenched disorder, one is mainly interested in averaging the free energy over the disorder, which amounts to averaging the log of the partition function Z . A well known technique for computing the average free energy is the replica method [24] based in results discussed in Ref. [25]. In this approach, a new system defined by k statistically independent replicas is introduced and the limit $k \rightarrow 0$ is taken. In this process, one has to understand the behavior of the symmetry group S_k as $k \rightarrow 0$. Moreover, special procedures are sometimes required for obtaining sound physical results. For example, the average free energy of the Sherrington-Kirkpatrick model [26], which is the infinite-range version of the Edwards-Anderson model, exhibits a (problematic) *negative entropy* at low temperatures, assuming a replica symmetric solution. The scheme of replica symmetry breaking was introduced to avoid this unphysical result. A step further was introduced by Parisi with another replica symmetry breaking solution, still using the replica formalism [27–29], by choosing a suitable ultrametric parametrization of the replica matrix in the computation of the average free energy. See also the Ref. [30]. This scheme describes many stable states with ultrametric structure in the phase space, in which the low-temperature phase consists of infinitely many pure thermodynamic states.

Despite the success in the application of the replica method in disordered systems, some authors consider that a mathematical rigorous derivation to support this procedure is still lacking [31–34].

It is therefore natural to ask whether there exists a mathematically rigorous method, based on the use of replicas, for computing the average free energy of systems with quenched disorder. In Ref. [35], Dotsenko considered an alternative approach where the summation of all integer moments of the partition function is used to evaluate the average free energy of the random energy model [36, 37]. Also a replica calculation using only the integer moments of the partition function have been considered in Ref. [38]. Some alternative approaches to the replica method are the dynamical approach [39] and the cavity method [40]. In the Ref. [41], the main concepts and methods of the cavity method is discussed.

In this paper we define a new mathematical object associated with systems with quenched disorder, namely, a complex function which, due to its similarities with zeta-functions, we will call distributional zeta-function. This terminology is in light with a probabilistic approach, where we introduce a probability distribution to define this zeta-function. We will show that the derivative of this function at the origin, which is well and uniquely defined, yields the average free energy of the underlying system with quenched disorder. Unlike the standard replica method, this procedure does not require derivation with respect to the (integer) moments of the partition function. Moreover, in our derivation the average free energy of the random $\lambda\varphi^4$ model is given by a series in which all the replicas contribute. This method is closely related to the use of spectral zeta-functions for evaluating the determinant of differential operators of different systems in Euclidean field theory [42–46] and for computing the renormalized vacuum energy of quantum field in the presence of boundaries. Although different global methods can be used to obtain the Casimir energy of quantum fields, as for example an exponential cut-off or an analytic regularization procedure [47–49], the spectral zeta-function method is powerful, elegant and widely used in quantum field theory [50].

The organization of this paper is as follows. In section II we discuss the replica field theory obtained from a quenched disordered field linearly coupled to the Euclidean scalar field. In section III we introduce the distributional zeta-function and compute the derivative of the distributional zeta-function at the origin in order to obtain the average free energy of the system with quenched disorder. Conclusions are given in section IV. In the appendix we briefly review the use of the spectral zeta-function method in Euclidean field theory. We use $\hbar = c = k_B = 1$.

2 The replica field theory for the disordered scalar model

Our aim is to use the Euclidean formalism of field theory to discuss classical statistical systems with disorder. We study a field theory with fluctuations around the mean field solution. Let us assume, for simplicity, that we have only a quenched disordered field linearly coupled to a Euclidean massive scalar field. In this case, one must obtain average values of extensive quantities. An establish technique for computing the average free energy is the replica method, which we briefly describe. The key point of this method is to compute integer moments of the partition function $\mathbb{E} Z^k$ and to use such information to calculate $\mathbb{E} \ln Z$. The replica method consists in the following steps. First, one constructs the (integer) k -th power of the partition function $Z^k = Z \times Z \times \dots \times Z$,

and interprets it as the partition function of a *new* system, formed of k statistically independent copies of the original system. Second, the expected value of the partition function's k -th power $Z_k = \mathbb{E} Z^k$ is computed by integrating over the disorder field on the new model. Finally, the average free energy is obtained using the formula $\mathbb{E} \ln Z = \lim_{k \rightarrow 0} \frac{Z_k - 1}{k}$, where Z_k for $0 < k < 1$ is derived from its values for k integer. Note that in Z_k , integration over the disorder field yields a system defined by k replicas which are no more statistically independent. The average value of the free energy in the presence of the quenched disorder is then obtained in the limit of a zero-component field theory, taking the limit $k \rightarrow 0$. Therefore, in this procedure it is necessary to understand the properties of the permutation group S_k , when $k \rightarrow 0$.

We start with the usual $\lambda\varphi^4$ Euclidean scalar field theory in the presence of a disorder field $h(x)$. In this setting, the functional integral $Z(h)$ is defined by [51]:

$$Z(h) = \int [d\varphi] \exp \left(-S + \int d^d x h(x) \varphi(x) \right), \quad (1)$$

where $S = S_0 + S_I$ is the action that usually describes a massive scalar field with the contributions S_0 and S_I given, respectively, by

$$S_0(\varphi) = \int d^d x \left(\frac{1}{2} (\partial\varphi)^2 + \frac{1}{2} m_0^2 \varphi^2(x) \right), \quad (2)$$

and

$$S_I(\varphi) = \int d^d x \frac{g_0}{4!} \varphi^4(x). \quad (3)$$

The term S_0 is the free field action while S_I , a non-Gaussian contribution, defines the interacting part. In Eq. (1), $[d\varphi]$ is a functional measure, formally given by $[d\varphi] = \prod_x d\varphi(x)$. The terms g_0 and m_0^2 are, respectively, the bare coupling constant and the bare mass square of the model. The term $h(x)$ is a quenched random field. A normalization factor appears in the definition of the gaussian functional integral, $\mathcal{N} = (\det(-\Delta + m_0^2))^{\frac{1}{2}}$, but in the following, as usual, we are absorbing this normalization factor in the functional measure. The symbol Δ denotes the Laplacian in \mathbb{R}^d .

One comment is in order. In the above formal functional integral formulation there are two kinds of random variables. The first one are the Euclidean fields $\varphi_i(x)$. These fields describes generalized random processes with zero mean and covariance $G_0(x-y, m_0) = \langle x | (-\Delta + m_0^2)^{-1} | y \rangle$. The second one, are the disordered field $h(x)$, in which the the absence of any gradient either makes them statistically independent in every point of the domain or, for fields that are not statistically independent for different points of the domain, the two-point correlation function is not defined in terms of gradients. We call these as uncorrelated and correlated disordered fields, respectively. We assume that the random variables characterizing the disorder exhibit no long-range correlations, therefore the probability distribution is written as

$$P(h) = p_0 \exp \left(-\frac{1}{2\sigma} \int d^d x (h(x))^2 \right). \quad (4)$$

The quantity σ is a small positive parameter associated with the disorder and p_0 is a normalization constant. In this case we have a delta correlated random field, with two-point correlation function $\mathbb{E}(h(x)h(y)) = \sigma\delta^d(x - y)$. In the next paragraph, we will obtain the replica partition function, $\mathbb{E} Z^k$, for this disordered model.

Let us construct $Z^k = Z \times Z \times \dots \times Z$. We interpret Z^k as the partition function of a new system, formed from k statistically independent copies of the original system. We have that the partition function Z^k of the replicated system is

$$(Z(h))^k = \int \prod_{i=1}^k [d\varphi_i] \exp\left(-\sum_{i=1}^k S(\varphi_i, h)\right). \quad (5)$$

Let us use the fact that h is a random variable with probability distribution $d\rho[h] = d[h]P(h)$. Averaging the free energy over the disorder we obtain that the replica partition function Z_k is given by

$$Z_k = \int \prod_{i=1}^k [d\varphi_i] \exp\left(-S_{eff}(\varphi_i, k)\right), \quad (6)$$

where the effective action $S_{eff}(\varphi_i, k)$ is

$$S_{eff}(\varphi_i, k) = \frac{1}{2} \sum_{i,j=1}^k \int d^d x \int d^d y \varphi_i(x) D_{ij}(x - y) \varphi_j(y) + \frac{\lambda}{4!} \sum_{i=1}^k \int d^d x \varphi_i^4(x). \quad (7)$$

In the above equation $D_{ij}(x - y) = D_{ij}(m_0, \sigma; x - y)$ where

$$D_{ij}(m_0, \sigma; x - y) = \left(\delta_{ij}(-\Delta + m_0^2) - \sigma\right) \delta^d(x - y). \quad (8)$$

The saddle-point equations of this model are given by

$$\left(-\Delta + m_0^2\right) \varphi_i(x) + \frac{\lambda}{3!} \varphi_i^3(x) = \sigma \sum_{j=1}^k \varphi_j(x). \quad (9)$$

Let us use a replica symmetric ansatz, i.e., suppose that $\varphi_i(x) = \varphi(x)$. For equal replicas, the saddle-point equations becomes a single equation given by

$$\left(-\Delta + m_0^2 - k\sigma\right) \varphi(x) + \frac{\lambda}{3!} \varphi^3(x) = 0. \quad (10)$$

In this case we must have $m_0^2 - k\sigma > 0$ to have a physical theory. The simplest case of identical replicas leads to the replica symmetry breaking situation.

Let us discuss first the perturbative expansion of the replica field theory. Starting from a gaussian theory, let us define $Z_{0k}(J_i)$ where we introduce external sources $J_i(x)$ which are smooth functions that we introduce to generate the correlation functions of the theory by functional derivatives. We have

$$Z_{0k}(J_i) = \int \prod_{i=1}^k [d\varphi_i] \exp \left(-\frac{1}{2} \sum_{i,j=1}^k \int d^d x \int d^d y \varphi_i(x) D_{ij}(x-y) \varphi_j(x) + \sum_{i=1}^k \varphi_i J_i \right), \quad (11)$$

where we have that

$$\sum_{i=1}^k \varphi_i J_i = \sum_{i=1}^k \int d^d x \varphi_i(x) J_i(x). \quad (12)$$

Performing the gaussian integrals we get

$$Z_{0k}(J_i) = \exp \left(\frac{1}{2} \sum_{i,j=1}^k \int d^d x \int d^d y J_i(x) [G_0]_{ij}(m_0, \sigma; x-y) J_j(y) \right), \quad (13)$$

where $[G_0]_{ij}(m_0, \sigma; x-y)$ is the two-point correlation function for the replica field theory. We have

$$\sum_{j=1}^k \int d^d z [G_0]_{ij}(m_0, \sigma; x-z) D_{jl}(z-y) = \delta_{il} \delta^d(x-y). \quad (14)$$

We can use $Z_{0k}(J_i)$ to obtain Z_k by functional derivatives. The Z_k will appear in the definition of the average free energy. We have

$$Z_k = \left[\exp \left(-S_I \left(\frac{\delta}{\delta J_i} \right) \right) Z_{0k}(J_i) \right] \Big|_{J_i=0}. \quad (15)$$

It is interesting to analyze the effective action in momentum space. Performing a Fourier transform we get

$$S_{eff}(\varphi_i, k) = \frac{1}{2} \sum_{i,j=1}^k \int \frac{d^d p}{(2\pi)^d} \varphi_i(p) [G_0]_{ij}^{-1} \varphi_j(-p) + \frac{\lambda}{4!} \sum_{i=1}^k \varphi_i^4, \quad (16)$$

where, in the quadratic part of $S_{eff}(\varphi_i, k)$, the quantity $[G_0]_{ij}^{-1}$ is the inverse of the two-point correlation function. In this tree-level approximation we have

$$[G_0]_{ij}^{-1}(p) = (p^2 + m_0^2) \delta_{ij} - \sigma. \quad (17)$$

To invert this matrix, let us express $[G_0]_{ij}^{-1}$ in terms of appropriate projector operators,

$$[G_0]_{ij}^{-1}(p) = (p^2 + m_0^2) \left(\delta_{ij} - \frac{1}{k} \right) + (p^2 + m_0^2 - k\sigma) \frac{1}{k}. \quad (18)$$

The projectors operators $(P_T)_{ij}$ and $(P_L)_{ij}$ are defined, respectively, as

$$(P_T)_{ij} = \delta_{ij} - \frac{1}{k} \quad (19)$$

and

$$(P_L)_{ij} = \frac{1}{k}. \quad (20)$$

Using the projectors operators we can write the two-point correlation function $[G_0]_{ij}(p)$ as

$$[G_0]_{ij}(p) = \frac{\delta_{ij}}{(p^2 + m_0^2)} + \frac{\sigma}{(p^2 + m_0^2)(p^2 + m_0^2 - k\sigma)}. \quad (21)$$

The first term in the right hand side of Eq. (21) is the bare contribution to the connected two-point correlation function; the second term is the contribution to the disconnected two-point correlation function, which becomes connected after averaging the disorder. The first term is characteristic of pure systems and the additional term, which becomes a squared Lorentzian term (for $k \rightarrow 0$), appears in these random field systems [52, 53].

In the next section we present a mathematical rigorous technique for computing the average free energy of a disordered system with quenched randomness using the integer moments of the partition function, without taking the limit of these integers to zero. The basic tool of this method is a complex function that we call the distributional zeta-function.

3 The distributional zeta-function in disordered systems

The purpose of this section is to present an alternative method to calculate the average free energy of a system with quenched disorder, using the replicas. Let \mathcal{M} be a Euclidean manifold. The fundamental objects here are the fields, a space $C^\infty(\mathcal{M}, \mathbb{R})$ of smooth functions on \mathcal{M} . Let $S : C^\infty(\mathcal{M}, \mathbb{R}) \rightarrow \mathbb{R}$ be an action functional. The Euclidean action functional associated with the model with disorder degrees of freedom can be written as

$$S(h, \varphi) = \int d^d x \left(\frac{1}{2} \varphi(x) (-\Delta + m_0^2) \varphi(x) + \frac{\lambda}{4!} \varphi^4(x) - h(x) \varphi(x) \right). \quad (22)$$

Again, the symbol Δ denotes the Laplacian in \mathbb{R}^d . The h -dependent free energy $F(h)$ is given by

$$F(h) = \ln \int [d\varphi] \exp \left(-S + \int d^d x h(x) \varphi(x) \right), \quad (23)$$

where, as above specified, S is the action including S_0 and S_I , i.e., the free and interaction contributions. The average free energy F_q is defined as

$$F_q = \int d\rho[h] F(h). \quad (24)$$

Recall that h is a random variable with probability distribution $d\rho[h] = d[h]P(h)$ with $d[h] = \prod_x dh(x)$. Direct use of these definitions yields

$$F_q = \int d[h]P(h) \ln Z(h). \quad (25)$$

One way to compute the average free energy is to use the standard replica method, briefly discussed in the previous section. Here we propose an alternative procedure to find the average free energy by means of a *distributional zeta-function*. Observe that if (X, \mathcal{A}, μ) is a measure space and $f : X \rightarrow (0, \infty)$ is measurable, one can define a generalized ζ -function

$$\zeta_{\mu, f}(s) = \int_X f(x)^{-s} d\mu(x) \quad (26)$$

for those $s \in \mathbb{C}$ such that $f^{-s} \in L^1(\mu)$, where in the above integral $f^{-s} = \exp(-s \log(f))$ is obtained using the principal branch of the logarithm. This formalism encompasses some well-known instances of zeta-functions for $f(x) = x$: if $X = \mathbb{N}$ and μ the counting measure we retrieve the Riemann zeta-function [54, 55]; instead, if μ counts only the prime numbers, we retrieve the prime zeta-function [56, 57] finally, if $X = \mathbb{R}$ and μ counts the eigenvalues of an elliptic operator, with their respective multiplicity, we obtain the spectral zeta-function [44]. Further extending this formalism to the case where $f(h) = Z(h)$ and $d\mu(h) = d[h]P(h)$ leads to the definition of the distributional zeta-function $\Phi(s)$,

$$\Phi(s) = \int d[h]P(h) \frac{1}{Z(h)^s} \quad (27)$$

for $s \in \mathbb{C}$, this function being defined in the region where the above integral converges. We will reproduce the steps of the use of the spectral zeta-function for computing functional determinants in field theory, proving that $\Phi(s)$ is well defined for $\text{Re}(s) \geq 0$, that $F_q = -(d/ds)\Phi(s)|_{s=0}$ and using this equality for computing a series approximation for the average free energy F_q .

Let us prove that $\Phi(s)$ is defined for $\text{Re}(s) \geq 0$. Since S_0 and S_I given by Eqs. (2) and (3) are even (as functions of φ), $Z(h) = Z(-h) = (Z(h) + Z(-h))/2$ and

$$Z(h) = \int [d\varphi] \exp(-S) \cosh \left(\int d^d x h(x) \varphi(x) \right). \quad (28)$$

Hence $Z(h) \geq Z(0)$ and, for $\text{Re}(s) \geq 0$, we have

$$\int d[h]P(h) \left| \frac{1}{Z(h)^s} \right| \leq \int d[h]P(h) \frac{1}{Z(0)^{\text{Re}(s)}} = \frac{1}{Z(0)^{\text{Re}(s)}} < \infty. \quad (29)$$

Therefore, the integral in Eq. (27) is well defined in the half complex plane $\text{Re}(s) \geq 0$ and Φ is also defined in this region, *without* resorting to analytic continuations.

Since $-(d/ds)Z(h)^{-s}|_{s=0+} = \ln Z(h)^1$, we can write

$$F_q = - \int d[h]P(h) \frac{d}{ds} \frac{1}{Z(h)^s} \Big|_{s=0+} = - \frac{d}{ds} \Phi(s) \Big|_{s=0+}, \quad (30)$$

where the second equality is justified by the fact that $Z(h) \geq Z(0)$ and an application of Lebesgue's dominated convergence theorem, if we interpret $d[h]P(h)$ as a measure. Observe that we obtained an analytic expression for F_q which, contrary to the standard replica method, *does not involve derivation of the (integer) moments of the partition function*. Direct use of Euler's integral representation for the gamma function give us

$$\frac{1}{Z(h)^s} = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} e^{-Z(h)t}, \quad \text{for } \text{Re}(s) > 0. \quad (31)$$

Although the above Mellin integral converges only for $\text{Re}(s) > 0$, as $Z(h) > 0$, we will show how to obtain from the above expression a formula for the free energy valid for $\text{Re}(s) \geq 0$. Substituting Eq. (31) in Eq. (27) we get

$$\Phi(s) = \frac{1}{\Gamma(s)} \int d[h]P(h) \int_0^\infty dt t^{s-1} e^{-Z(h)t}. \quad (32)$$

We already know that the distributional zeta function $\Phi(s)$ is defined for $\text{Re}(s) \geq 0$. Now we will use the above expression for computing its derivative at $s = 0$ by analytic tools. We assume at principle the commutativity of the following operations, disorder average, differentiation, integration if necessary.

From the above discussion, it follows that the average free energy F_q can be written as

$$F_q = - \frac{d}{ds} \frac{1}{\Gamma(s)} \int d[h]P(h) \int_0^\infty dt t^{s-1} e^{-Z(h)t} \Big|_{s=0+}. \quad (33)$$

To continue, take $a > 0$ and write $\Phi = \Phi_1 + \Phi_2$ where

$$\Phi_1(s) = \frac{1}{\Gamma(s)} \int d[h]P(h) \int_0^a dt t^{s-1} e^{-Z(h)t} \quad (34)$$

¹ $(d/ds)f|_{s=0+}$ stands for $\lim_{s \rightarrow 0+} \frac{f(s)-f(0)}{s}$, whenever this limit exists

and

$$\Phi_2(s) = \frac{1}{\Gamma(s)} \int d[h] P(h) \int_a^\infty dt t^{s-1} e^{-Z(h)t}. \quad (35)$$

The average free energy can be written as

$$F_q = -\frac{d}{ds} \Phi_1(s)|_{s=0+} - \frac{d}{ds} \Phi_2(s)|_{s=0}. \quad (36)$$

The integral $\Phi_2(s)$ defines an analytic function defined in the whole complex plane. In the innermost integral in $\Phi_1(s)$ the series representation for the exponential converges uniformly (for each h), so that we can reverse the order on integration and summation to obtain

$$\Phi_1(s) = \int d[h] P(h) \frac{1}{\Gamma(s)} \sum_{k=0}^{\infty} \frac{(-1)^k a^{k+s}}{k!(k+s)} Z(h)^k. \quad (37)$$

The term $k = 0$ in the above summations contain a removable singularity at $s = 0$ because $\Gamma(s)s = \Gamma(s+1)$, so that we can write

$$\Phi_1(s) = \frac{a^s}{\Gamma(s+1)} + \frac{1}{\Gamma(s)} \sum_{k=1}^{\infty} \frac{(-1)^k a^{k+s}}{k!(k+s)} \mathbb{E} Z^k, \quad (38)$$

an expression valid for $\text{Re}(s) \geq 0$. The function $\Gamma(s)$ has a pole in at $s = 0$ with residue 1, therefore

$$-\frac{d}{ds} \Phi_1(s)|_{s=0+} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} a^k}{k!k} \mathbb{E} Z^k + f(a), \quad (39)$$

where

$$f(a) = -\frac{d}{ds} \left(\frac{a^s}{\Gamma(s+1)} \right) |_{s=0} = -(\ln(a) + \gamma) \quad (40)$$

and γ is Euler's constant $0.577 \dots$. We have just shown that the contribution of $-(d/ds)\Phi_1(s)|_{s=0+}$ to the average free energy (see Eq. (39)) can be written as a series in which all integer moments of the partition function appear. Although such a representation cannot be obtained, in general, for the contribution coming from $-(d/ds)\Phi_2(s)|_{s=0}$, we will show how to bound this contribution. Using again the fact that $\Gamma(s)$ has a pole at $s = 0$ with residue 1 and taking the derivative of the product that defines Φ_2 in Eq. (35), we conclude that

$$-\frac{d}{ds} \Phi_2(s)|_{s=0} = -\int d[h] P(h) \int_a^\infty \frac{dt}{t} e^{-Z(h)t} = R(a). \quad (41)$$

Therefore, using again the inequality $Z(h) \geq Z(0)$ we obtain the bound

$$\left| -\frac{d}{ds} \Phi_2(s)|_{s=0} \right| \leq \int d[h] P(h) \int_a^\infty \frac{dt}{t} e^{-Z(0)t} \leq \frac{1}{Z(0)a} \exp(-Z(0)a). \quad (42)$$

In conclusion, using the distributional zeta-function we are able to represent the average free energy as

$$F_q = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} a^k}{k!k} \mathbb{E} Z^k - (\ln(a) + \gamma) + R(a), \quad |R(a)| \leq \frac{1}{Z(0)a} \exp(-Z(0)a). \quad (43)$$

Observe that we cannot take the limit $a \rightarrow \infty$, because in this case the above series become meaningless. However, the contribution of $R(a)$ to the free energy can be made as small as desired, taking a large enough. A case of special interest is $a = 1$, because with this choice the average free energy can be written as

$$F_q = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!k} \mathbb{E} Z^k - \gamma + R(1), \quad |R(1)| \leq \frac{1}{Z(0)} \exp(-Z(0)). \quad (44)$$

Defining a distributional zeta-function and using analytic tools, we showed that all replicas contribute to the average value of the free energy of the system. Using Eq. (15) and Eq. (44) we can write the average free energy in terms of the vacuum to vacuum diagrams.

The main result of this paper is Eq. (44). Instead of the limit to zero replicas as in the usual procedure, we obtained an expression for the average free energy as a series where all the replicas contribute plus another term which can not be represented by a series on the replicas partition function. For $Z(0)$ bounded away from zero, the second term contribution's can be made as small as desired. Note that the representation of the average free energy just by a series on the the replicas partition functions would not describe the loss of analyticity on the average free energy in the $Z(h) \rightarrow 0$ phase transitions.

Two comments are in order. First, in those phase transitions where $Z(h) \rightarrow 0$, the divergence of the free energy comes from $R(a) = -(d/ds)\Phi_2(s)$ as revealed by direct inspection of Eq. (41); the contribution to the free energy due to the series expansion captures its non-analytic behaviour when $Z(h) \rightarrow \infty$. Second, in the replica field theory the symmetry group is S_k (the permutation group of k elements). One advantage of the distributional zeta-function technique is that it is not necessary to understand the properties of the permutation group when $k \rightarrow 0$. Let us discuss the replica symmetric ansatz, i.e., suppose that $\varphi_i(x) = \varphi(x)$. For the case of identical replicas, the saddle-point equations make sense for $k < \frac{m_0^2}{\sigma}$. Therefore, in this case one could justifies this replica symmetric solution. For $k \geq \frac{m_0^2}{\sigma}$, we must define a shifted field $\phi(x) = \varphi(x) - v$, for $v = (\frac{k\sigma - m_0^2}{6\lambda})^{1/2}$. In terms of this new field $\phi(x)$, we get a positive mass squared with new self-interaction vertices $\phi^3(x)$ and $\phi(x)^4$. Therefore the symmetry between the replicas is broken using the results of the saddle-point equations of the model.

4 Conclusions

There is a growing interest in disordered systems in physics and many areas beyond physics. In statistical mechanics, the random magnetic field Ising model has been investigated in the last forty years. The hamiltonian of the random field Ising model is analogous to the one of the classical Ising model, but allowing for a quenched random magnetic field. This model can be analyzed using a collective description instead of dealing with individual elements. The hamiltonian of the discrete model can be replaced by an effective hamiltonian of the Landau-Ginzburg model, where the order parameter is a continuous field $\varphi(x)$ coupled to a quenched random field. The usual approach to study such disordered systems is to transform the random problem into a translational invariant one. For quenched disorder, one is mainly interested in averaging the free energy over the disorder, which amounts to averaging the log of the partition function Z , the connected vacuum to vacuum diagrams. The replica method is a powerful tool used to calculate the free energy of systems with quenched disorder. Despite the absence of a mathematically rigorous derivation, the standard replica method provides correct results in many situations. As we discussed, it is natural to ask if it is possible to find a mathematically rigorous derivation which legitimates the use of the replica partition functions for computing the average free energy of the system. In this paper we present a rigorous derivation of the average free energy in the continuous version of the random field Ising model, as the sum of a series using the replica partition functions plus a contribution which, far from phase transitions, can be made as small as desired.

In order to find the average free energy we define a distributional zeta-function. The derivative of the distributional zeta-function at $s = 0$ yields the average free energy. Making use of the Mellin transform and analytic continuation, it is possible to obtain a series representation for the average free energy where all the integer moments of the partition function of the model contribute. Our method is closely related to the use of spectral zeta-functions for evaluating the determinant of elliptic differential operators in Euclidean field theory. This technique also has been used to calculate the renormalized vacuum energy of quantum field in the presence of boundaries.

The distributional zeta-function technique is a mathematically rigorous method for computing the average free energy using the replica partition functions. Further validation of the distributional zeta-function technique in the Sherrington-Kirkpatrick [26] and also the random energy model is the natural continuation of the present work. In the random energy model it is known that for $k < 1$ there is a phase transition at some critical temperature, but high and low temperature regimes exchange their roles. Also in the Sherrington-Kirkpatrick model, after integrating over the random spin couplings and using the saddle-point approximation, the average free energy density is written in terms of an order parameter $k \times k$ matrix Q_{ij} . The stability of the model implies that the Hessian matrix $\frac{\partial^2 F}{\partial Q_{ij} \partial Q_{kl}}, H$ has positive eigenvalues. For example, in the subspace of the matrix satisfying $Q_{ij} = q$, the condition above leads to $\frac{1}{k} Tr Q^2 = (k - 1)q^2$, which is always valid for $k > 1$. In the case of the Sherrington-Kirkpatrick model, at some stage of the calculations we must use the saddle-point equations to extremize the free energy with respect to some variable. As stressed by Dotsenko in Ref. [35], it is not clear that this condition of extremizing the free energy can be used in this formalism, where all the moments of the partition function appear in the average free energy.

There are many papers studying finite size-effects in Euclidean field theory [58–62]. Suppose

a Euclidean field theory with a $(\lambda\varphi^4)$ self-interaction defined in a d -dimensional Euclidean space. Assume that the domain where the field is defined is $S^1 \times \mathbb{R}^{d-1}$. For very small radius, since there is a size-dependent renormalized mass square, the system is in the disordered phase. Increasing the radius of the compactified dimension, we obtain the critical size where a second order phase transition occurs [63, 64]. A complete discussion of finite size effects in phase transitions can be found in [65]. For the reader interested in the study of finite size effects in random systems, see Refs. [66–69].

Recently, it was discussed a phase transition in the finite-size disordered $\lambda\varphi^4$ model, using the standard replica method. The authors considered the scalar model system at zero temperature defined in a space with periodic boundary conditions in one space dimension. It was shown that there is a critical length where the system develop a second-order phase transition, where the system presents long-range correlations with power law decay [70]. As an application of our approach, it would be interesting to discuss finite-size effects in one disordered $\lambda\varphi^4$ model in a d -dimensional Euclidean space using this distributional zeta-function technique. Does the renormalized mass obtained from the distributional zeta-function coincide with the renormalized mass obtained by the usual replica method where the average free energy is obtained from a zero component field theory? This problem also will be subject of future investigations.

5 Acknowledgments

We would like to thanks G. Parisi and S. Fulling for encouragements. We would like to acknowledge also V. Dotsenko, G. Krein, E. Curado, T. Micklitz, S. Alves Dias, F. Nobre and G. Menezes for the fruitful discussions. This paper was supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico do Brazil (CNPq).

A The spectral zeta function in Euclidean field theory

Here we present a new mathematical rigorous technique for computing the average free energy of a disordered system with quenched randomness, using the replicas. The basic tool of this technique is a distributional zeta-function. In this appendix we would like to discuss the similarities of our approach with the spectral zeta-function regularization in Euclidean field theory [44, 45].

If λ_k is a sequence of non-zero complex numbers, we define the zeta regularized product of these numbers as

$$\prod_k \lambda_k = e^{-\frac{d}{ds} \zeta(s)|_{s=0}}, \quad (\text{A.1})$$

where

$$\zeta(s) = \sum_k \frac{1}{\lambda_k^s} \text{ for } \text{Re}(s_0) > 0, \quad (\text{A.2})$$

is the zeta-function associated with the sequence λ_k . If λ_k is the sequence of the positive eigenvalues of the Laplacian on a manifold, then the zeta regularized product is the determinant of the

Laplacian. The free energy of a system with this eigenvalues in this case is

$$F = \frac{1}{2} \frac{d}{ds} \zeta_D(s)|_{s=0}. \quad (\text{A.3})$$

In order to obtain the free energy of a system, we have to show that $\zeta_D(s)$ has a meromorphic continuation with at most simple poles, to the half-plane containing the origin and analytic at the origin. We have scaling properties, i.e.,

$$-\frac{d}{ds} \zeta_{\mu D}(s)|_{s=0} = -\frac{d}{ds} \zeta_D(s)|_{s=0} + \ln \mu \zeta_D(s)|_{s=0}. \quad (\text{A.4})$$

Note that there are sequences of numbers which are not zeta regularized, where the free energy is not defined [71, 72]. Suppose that the sequence of numbers is the sequence of the prime numbers. In this case the prime zeta function $P(s)$, $s = \sigma + i\tau$, for $\sigma, \tau \in \mathbb{R}$ is defined as

$$P(s) = \sum_p p^{-s}, \quad \text{Re}(s) > 1. \quad (\text{A.5})$$

Introducing the Möbius function $\mu(n)$ [73] we can write

$$P(s) = \sum_k \frac{\mu(k)}{k} \ln \zeta(ks), \quad (\text{A.6})$$

where $\zeta(s)$ is the Riemann zeta function [54]. The line $\text{Re}(s) = 0$ is a natural boundary of $P(s)$. Therefore the prime zeta function can be analytically extended only in the strip $0 < \sigma \leq 1$ [56, 57]. Since we need to compute the derivative of the spectral zeta function at $s = 0$, the free energy is not defined. The sequence of primes numbers is not zeta-regularized.

In order to discuss the similarities of our approach with the spectral zeta-function regularization in Euclidean field theory, let us discuss the heat-kernel on Euclidean manifolds. Let \mathcal{M} be an arbitrary smooth connected d -dimensional compact Riemannian manifold. The Riemannian structure allow to introduce on \mathcal{M} volumes of all dimensions. The Riemannian d -volume is defined as $d\mu = \sqrt{g}dx^1 \dots dx^d$. Now consider the following initial boundary problem in $(0, \infty) \times \mathbb{R}^d$. We have

$$\frac{\partial}{\partial t} u(t, x) = \Delta u(t, x), \quad (\text{A.7})$$

with the initial condition

$$u(0, x) = f(x), \quad (\text{A.8})$$

and also

$$u(t, x)|_{x \in \partial \Omega} = 0, \quad (\text{A.9})$$

where the symbol Δ denotes the Laplacian in \mathbb{R}^d . The evolution equation has the solution

$$u(t, x) = e^{t\Delta_\Omega} f(x). \quad (\text{A.10})$$

Suppose that the function $f(x)$ is $f \in L^2(\Omega)$. The standard technique of the spectral theory of elliptic operators implies that there exists a orthonormal basis $[\phi_k]$ in $L^2(\Omega)$ such that ϕ_k is a eigenfunction of $-\Delta$ in Ω with the eigenvalue $\lambda_k = \lambda_k(\Omega)$, and $0 \leq \lambda_1 \leq \lambda_2 \dots \lambda_k \rightarrow \infty$, when $k \rightarrow \infty$. Therefore we can write the following series representation

$$f(x) = \sum_{k=1}^{\infty} a_k \phi_k(x), \quad (\text{A.11})$$

where

$$a_k = \int_{\Omega} d\mu(y) f(y) \phi_k(y). \quad (\text{A.12})$$

We can write

$$u(t, x) = \int_{\Omega} p_{\Omega}(t, x, y) f(y) d\mu(y), \quad (\text{A.13})$$

where

$$p_{\Omega}(t, x, y) = \sum_{k=1}^{\infty} e^{-\lambda_k(\Omega)} \phi_k(x) \phi_k(y). \quad (\text{A.14})$$

We denote $p_{\Omega}(t, x, y)$ the heat-kernel of Ω . The operator $e^{t\Delta_{\Omega}}$ has the integral kernel $p_{\Omega}(t, x, y)$. We have

$$\int_{\Omega} d\mu(y) p_{\Omega}(t, y, y) = \sum_{k=1}^{\infty} e^{-t\lambda_k(\Omega)} \quad (\text{A.15})$$

Recall that the Mellin transform of $f : \mathbb{R}_+ \rightarrow \mathbb{C}$ is defined as

$$M[f; s] = \int_0^{\infty} \frac{dx}{x} f(x) x^s \quad (\text{A.16})$$

for those $s \in \mathbb{C}$ such that $f(x) x^s \in L^1(\mathbb{R}_+)$. For example, if $f(x) = e^{-x}$ then the Mellin transform is the gamma function, i.e.,

$$\Gamma(s) = \int_0^{\infty} dt t^{s-1} e^{-t}, \quad \text{for } \text{Re}(s) > 0. \quad (\text{A.17})$$

We can write that the free energy is given by

$$F = \frac{1}{2} \frac{d}{ds} \left[\frac{M[\int d\mu(y) p_{\Omega}(t, y, y), s]}{M[e^{-t}, s]} \right] \Big|_{s=0}. \quad (\text{A.18})$$

Our mathematical approach to make rigorous the replica method using a distributional zeta-function can be compared with the spectral zeta-function method widely used in quantum field

theory. Using the Eq. (33) the average free energy of the continuous version for the random field Ising model can be written as

$$F_q = -\frac{d}{ds} \left[\frac{M[\int d[h]P(h)e^{-Z(h)t}, s]}{M[e^{-t}, s]} \right]_{s=0^+}. \quad (\text{A.19})$$

Note the similarities between the Eq. (A.18) and Eq. (A.19).

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